

Effective Capacity and Randomness of Closed Sets

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We investigate the connection between measure and capacity for the space \mathcal{C} of nonempty closed subsets of $2^{\mathbb{N}}$. For any computable measure μ^* , a computable capacity \mathcal{T} may be defined by letting $\mathcal{T}(Q)$ be the measure of the family of closed sets K which have nonempty intersection with Q . We prove an effective version of Choquet's capacity theorem by showing that every computable capacity may be obtained from a computable measure in this way. We establish conditions that characterize when the capacity of a random closed set equals zero or is > 0 . We construct for certain measures an effectively closed set with positive capacity and with Lebesgue measure zero.

1 Introduction

The study of algorithmic randomness has been an active area of research in recent years. The basic problem is to quantify the randomness of a single real number. Here we think of a real $r \in [0, 1]$ as an infinite sequence of 0's and 1's, i.e. as an element in $2^{\mathbb{N}}$. There are three basic approaches to algorithmic randomness: the measure-theoretic approach of Martin-Löf tests, the incompressibility approach of Kolmogorov complexity, and the betting approach in terms of martingales. All three approaches have been shown to yield the same notion of (algorithmic) randomness. The present paper will consider only the measure-theoretic approach. A real x is Martin-Löf random if for any effective sequence S_1, S_2, \dots of c. e. open sets with $\mu(S_n) \leq 2^{-n}$, $x \notin \bigcap_n S_n$. For background and history of algorithmic randomness we refer to [9, 15].

In a series of recent papers [2, 3, 4, 5], G. Barmpalias, S. Dashti, R. Weber and the authors have defined a notion of (algorithmic) randomness for closed sets and continuous functions on $2^{\mathbb{N}}$. Some definitions are needed. For a finite string $\sigma \in \{0, 1\}^n$, let $|\sigma| = n$. For two strings σ, τ , say that τ *extends* σ and write $\sigma \sqsubseteq \tau$ if $|\sigma| \leq |\tau|$ and $\sigma(i) = \tau(i)$ for $i < |\sigma|$. For $x \in 2^{\mathbb{N}}$, $\sigma \sqsubset x$ means that $\sigma(i) = x(i)$ for $i < |\sigma|$. Let $\sigma \frown \tau$ denote the concatenation of σ and τ and let $\sigma \frown i$ denote $\sigma \frown (i)$ for $i = 0, 1$. Let $x \upharpoonright n = (x(0), \dots, x(n-1))$. Two reals x and y may be coded together into $z = x \oplus y$, where $z(2n) = x(n)$ and $z(2n+1) = y(n)$ for all n . For a finite string σ , let $I(\sigma)$ denote $\{x \in 2^{\mathbb{N}} : \sigma \sqsubset x\}$. We shall call $I(\sigma)$ the *interval* determined by σ . Each such interval is a clopen set and the clopen sets are just finite unions of intervals. We let \mathcal{B} denote the Boolean algebra of clopen sets.

Now a closed set P may be identified with a tree $T_P \subseteq \{0, 1\}^*$ where $T_P = \{\sigma : P \cap I(\sigma) \neq \emptyset\}$. Note that T_P has no dead ends. That is, if $\sigma \in T_P$, then either $\sigma \frown 0 \in T_P$ or $\sigma \frown 1 \in T_P$. set of infinite paths through T . It is well-known that $P \subseteq 2^{\mathbb{N}}$ is a closed set if and only if $P = [T]$ for some tree T . P is a Π_1^0

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class, or an effectively closed set, if $P = [T]$ for some computable tree T ; equivalently T_P is a Π_1^0 set. The complexity of the closed set P is generally identified with that of T_P . Thus P is said to be a Π_2^0 closed set if T_P is Π_2^0 ; in this case $P = [T]$ for some Δ_2^0 tree T . The complement of a Π_1^0 class is sometimes called a c.e. open set. We remark that if P is a Π_1^0 class, then T_P is a Π_1^0 set, but it is not, in general, computable. There is a natural effective enumeration P_0, P_1, \dots of the Π_1^0 classes and thus an enumeration of the c.e. open sets. Thus we can say that a sequence S_0, S_1, \dots of c.e. open sets is *effective* if there is a computable function, f , such that $S_n = 2^{\mathbb{N}} - P_{f(n)}$ for all n . For a detailed development of Π_1^0 classes, see [7].

It was observed in [3] that there is a natural isomorphism between the space \mathcal{C} of nonempty closed subsets of $\{0, 1\}^{\mathbb{N}}$ and the space $\{0, 1, 2\}^{\mathbb{N}}$ (with the product topology) defined as follows. Given a nonempty closed $Q \subseteq 2^{\mathbb{N}}$, let $T = T_Q$ be the tree without dead ends such that $Q = [T]$. Let $\sigma_0, \sigma_1, \dots$ enumerate the elements of T in order, first by length and then lexicographically. We then define the code $x = x_Q = x_T$ by recursion such that for each n , $x(n) = 2$ if both $\sigma_n \frown 0$ and $\sigma_n \frown 1$ are in T , $x(n) = 1$ if $\sigma_n \frown 0 \notin T$ and $\sigma_n \frown 1 \in T$, and $x(n) = 0$ if $\sigma_n \frown 0 \in T$ and $\sigma_n \frown 1 \notin T$. For a finite tree $T \subseteq \{0, 1\}^{\leq n}$, the finite code ρ_T is similarly defined, ending with $\rho_T(k)$ where σ_k is the lexicographically last element of $T \cap \{0, 1\}^n$.

We defined in [3] a measure μ^* on the space \mathcal{C} of closed subsets of $2^{\mathbb{N}}$ as follows.

$$\mu^*(\mathcal{X}) = \mu(\{x_Q : Q \in \mathcal{X}\}) \quad (1)$$

for any $\mathcal{X} \subseteq \mathcal{C}$ and μ is the standard measure on $\{0, 1, 2\}^{\mathbb{N}}$. Informally this means that given $\sigma \in T_Q$, there is probability $\frac{1}{3}$ that both $\sigma \frown 0 \in T_Q$ and $\sigma \frown 1 \in T_Q$ and, for $i = 0, 1$, there is probability $\frac{1}{3}$ that only $\sigma \frown i \in T_Q$. In particular, this means that $Q \cap I(\sigma) \neq \emptyset$ implies that for $i = 0, 1$, $Q \cap I(\sigma \frown i) \neq \emptyset$ with probability $\frac{2}{3}$.

Brodhead, Cenzer, and Dashti [3] defined a closed set $Q \subseteq 2^{\mathbb{N}}$ to be (Martin-Löf) random if x_Q is (Martin-Löf) random. Note that the equal probability of $\frac{1}{3}$ for the three cases of branching allows the application of Schnorr's theorem that Martin-Löf randomness is equivalent to prefix-free Kolmogorov randomness. Then in [3, 4], the following results are proved. No Π_1^0 class is random but there is a random Δ_2^0 closed set. Every random closed set contains a random member but not every member is random. Every random real belongs to some random closed set. Every random Δ_2^0 closed set contains a random Δ_2^0 member. Every random closed set is perfect and contains no computable elements (in fact, it contains no n -c.e. elements). Every random closed set has measure 0. A random closed set is a specific type of random recursive construction, as studied by Graf, Mauldin and Williams [10]. McLinden and Mauldin [13] showed that the Hausdorff dimension of a random closed set is $\log_2(4/3)$.

Just as an effectively closed set in $2^{\mathbb{N}}$ may be viewed as the set of infinite paths through a computable tree $T \subseteq \{0, 1\}^*$, an algorithmically random closed set in $2^{\mathbb{N}}$ may be viewed as the set of infinite paths through an algorithmically random tree T . Diamondstone and Kjos-Hanssen [12, 11] give an alternative definition of random closed sets according to the Galton-Watson distribution and show that this definition produces the same family of algorithmically random closed sets. The effective Hausdorff dimension of members of random closed sets is studied in [12].

In the present paper we will examine the notion of computable capacity and its relation to computable measures on the space \mathcal{C} of nonempty closed sets. In section two, we present a family of computable measures on \mathcal{C} and show how they induce capacities. We define the notion of computable capacity and present an effective version of Choquet's theorem that every capacity can be obtained from a measure μ^* on the space of closed sets. The main theorem of section three gives conditions under which the capacity $\mathcal{T}(Q)$ of a μ^* -random closed set Q is either equal to 0 or > 0 . We also construct a Π_1^0 class with Lebesgue measure zero but with positive capacity, for each capacity of a certain type.

2 Computable Measure and Capacity on the Space of Closed Sets

In this section, we describe the hit-or-miss topology on the space \mathcal{C} of closed sets, we define certain probability measures μ_d on the space $\{0, 1, 2\}^{\mathbb{N}}$ and the corresponding measures μ_d^* on the homeomorphic space \mathcal{C} . We present an effective version of Choquet's theorem connecting measure and capacity.

The standard (*hit-or-miss*) topology [8] (p. 45) on the space \mathcal{C} of closed sets is given by a sub-basis of sets of two types, where U is any open set in $2^{\mathbb{N}}$.

$$V(U) = \{K : K \cap U \neq \emptyset\}; \quad W(U) = \{K : K \subseteq U\}$$

Note that $W(\emptyset) = \{\emptyset\}$ and that $V(2^{\mathbb{N}}) = \mathcal{C} \setminus \{\emptyset\}$, so that \emptyset is an isolated element of \mathcal{C} under this topology. Thus we may omit \emptyset from \mathcal{C} without complications.

A basis for the hit-or-miss topology may be formed by taking finite intersections of the basic open sets. We want to work with the following simpler basis. For each n and each finite tree $A \subseteq \{0, 1\}^{\leq n}$, let

$$U_A = \{K \in \mathcal{C} : (\forall \sigma \in A)(K \cap I(\sigma) \neq \emptyset) \text{ \& } (\forall \sigma \notin A)(K \cap I(\sigma) = \emptyset)\}.$$

That is,

$$U_A = \{K \in \mathcal{C} : T_K \cap \{0, 1\}^{\leq n} = A\}.$$

Note that the sets U_A are in fact clopen. That is, for any tree $A \subseteq \{0, 1\}^{\leq n}$, define the tree $A' = \{\sigma \in \{0, 1\}^{\leq n} : (\exists \tau \in \{0, 1\}^n \setminus A) \sigma \sqsubseteq \tau\}$. Then $U_{A'}$ is the complement of U_A .

For any finite n and any tree $T \subseteq \{0, 1\}^{\leq n}$, define the clopen set $[T] = \bigcup_{\sigma \in T} I(\sigma)$. Then $K \cap [T] \neq \emptyset$ if and only if there exists some $A \subseteq \{0, 1\}^{\leq n}$ such that $K \in U_A$ and $A \cap T \neq \emptyset$. That is,

$$V([T]) = \bigcup \{U_A : A \cap T \neq \emptyset\}.$$

Similarly, $K \subseteq [T]$ if and only if there exists some $A \subseteq \{0, 1\}^n$ such that $K \in U_A$ and $A \subseteq T$. That is,

$$W([T]) = \bigcup \{U_A : A \subseteq T\}.$$

The following lemma can now be easily verified.

Lemma 2.1. *The family of sets $\{U_A : A \subseteq \{0, 1\}^{\leq n} \text{ a tree}\}$ is a basis of clopen sets for the hit-or-miss topology on \mathcal{C} .*

Recall the mapping from \mathcal{C} to $\{0, 1, 2\}^{\mathbb{N}}$ taking Q to x_Q . It can be shown that this is in fact a homeomorphism. (See Axon [1] for details.) Let \mathcal{B}^* be the family of clopen sets in \mathcal{C} ; each set is a finite union of basic sets of the form U_A and thus \mathcal{B}^* is a computable atomless Boolean algebra.

Proposition 2.2. *The space \mathcal{C} of nonempty closed subsets of $2^{\mathbb{N}}$ is homeomorphic to the space $\{0, 1, 2\}^{\mathbb{N}}$. Furthermore, the corresponding map from \mathcal{B} to \mathcal{B}^* is a computable isomorphism.*

Next we consider probability measures μ on the space $\{0, 1, 2\}^{\mathbb{N}}$ and the corresponding measures μ^* on \mathcal{C} induced by μ .

A probability measure on $\{0, 1, 2\}^{\mathbb{N}}$ may be defined as in [16] from a function $d : \{0, 1, 2\}^* \rightarrow [0, 1]$ such that $d(\lambda) = 1$ and, for any $\sigma \in \{0, 1, 2\}^*$,

$$d(\sigma) = \sum_{i=0}^2 d(\sigma \frown i).$$

The corresponding measure μ_d on $\{0, 1, 2\}^{\mathbb{N}}$ is then defined by letting $\mu_d(I(\sigma)) = d(\sigma)$. Since the intervals $I(\sigma)$ form a basis for the standard product topology on $\{0, 1, 2\}^{\mathbb{N}}$, this will extend to a measure on all Borel sets. If d is computable, then μ_d is said to be computable. The measure μ_d is said to be *nonatomic* or *continuous* if $\mu_d(\{x\}) = 0$ for all $x \in \{0, 1, 2\}^{\mathbb{N}}$. We will say that μ_d is *bounded* if there exist bounds $b, c \in (0, 1)$ such that, for any $\sigma \in \{0, 1, 2\}^*$ and $i \in \{0, 1, 2\}$,

$$b \cdot d(\sigma) < d(\sigma \frown i) < c \cdot d(\sigma).$$

It is easy to see that any bounded measure must be continuous. We will say that the measure μ_d is *regular* if there exist constants b_0, b_1, b_2 with $b_0 + b_1 + b_2 = 1$ such that for all σ and for $i \leq 2$, $d(\sigma \frown i) = b_i d(\sigma)$.

Now let μ_d^* be defined by

$$\mu_d^*(\mathcal{X}) = \mu_d(\{x_Q : Q \in \mathcal{X}\}).$$

Let us say that a measure μ^* on \mathcal{C} is computable if the restriction of μ^* to \mathcal{B}^* is computable.

Proposition 2.3. *For any computable d , the measure μ_d^* is a computable measure on \mathcal{C} .*

Proof. For any tree $A \subseteq \{0, 1\}^{\leq n}$, it is easy to see that

$$K \in U_A \iff \rho_A \sqsubset x_K,$$

so that $\mu_d^*(U_A) = \mu_d(I(\rho_A))$. □

We are now ready to define capacity. For details on capacity and random set variables, see [14].

Definition 2.4. *A capacity on \mathcal{C} is a function $\mathcal{T} : \mathcal{C} \rightarrow [0, 1]$ with $\mathcal{T}(\emptyset) = 0$ such that*

(i) *\mathcal{T} is monotone increasing, that is,*

$$Q_1 \subseteq Q_2 \longrightarrow \mathcal{T}(Q_1) \leq \mathcal{T}(Q_2).$$

(ii) *\mathcal{T} has the alternating of infinite order property, that is, for $n \geq 2$ and any $Q_1, \dots, Q_n \in \mathcal{C}$*

$$\mathcal{T}\left(\bigcap_{i=1}^n Q_i\right) \leq \sum_{I \subseteq \{1, 2, \dots, n\}} \{(-1)^{|I|+1} \mathcal{T}\left(\bigcup_{i \in I} Q_i\right) : \emptyset \neq I \subseteq \{1, 2, \dots, n\}\}.$$

(iii) *If $Q = \bigcap_n Q_n$ and $Q_{n+1} \subseteq Q_n$ for all n , then $\mathcal{T}(Q) = \lim_{n \rightarrow \infty} \mathcal{T}(Q_n)$.*

We will also assume, unless otherwise specified, that the capacity $\mathcal{T}(2^{\mathbb{N}}) = 1$.

We will say that a capacity \mathcal{T} is computable if it is computable on the family of clopen sets, that is, if there is a computable function F from the Boolean algebra \mathcal{B} of clopen sets into $[0, 1]$ such that $F(B) = \mathcal{T}(B)$ for any $B \in \mathcal{B}$.

Define $\mathcal{T}_d(Q) = \mu_d^*(V(Q))$. That is, $\mathcal{T}_d(Q)$ is the probability that a randomly chosen closed set meets Q . Here is the first result connecting measure and capacity.

Theorem 2.5. *If μ_d^* is a (computable) probability measure on \mathcal{C} , then \mathcal{T}_d is a (computable) capacity.*

Proof. Certainly $\mathcal{T}_d(\emptyset) = 0$. The alternating property follows by basic probability. For (iii), suppose that $Q = \bigcap_n Q_n$ is a decreasing intersection. Then by compactness, $Q \cap K \neq \emptyset$ if and only if $Q_n \cap K \neq \emptyset$ for all n . Furthermore, $V(Q_{n+1}) \subseteq V(Q_n)$ for all n . Thus

$$\mathcal{T}_d(Q) = \mu_d^*(V(Q)) = \mu_d^*(\bigcap_n V(Q_n)) = \lim_n \mu_d^*(V(Q_n)) = \lim_n \mathcal{T}_d(Q_n).$$

If d is computable, then \mathcal{T}_d may be computed as follows. For any clopen set $I(\sigma_1) \cup \dots \cup I(\sigma_k)$ where each $\sigma_i \in \{0, 1\}^n$, we compute the probability distribution for all trees of height n and add the probabilities of those trees which contain one of the σ_i . □

Choquet's Capacity Theorem states that any capacity \mathcal{T} is determined by a measure, that is $\mathcal{T} = \mathcal{T}_d$ for some d . See [14] for details. We now give an effective version of Choquet's theorem.

Theorem 2.6 (Effective Choquet Capacity Theorem). *If \mathcal{T} is a computable capacity, then there is a computable measure μ_d^* on the space of closed sets such that $\mathcal{T} = \mathcal{T}_d$.*

Proof. Given the values $\mathcal{T}(U)$ for all clopen sets $I(\sigma_1) \cup \dots \cup I(\sigma_k)$ where each $\sigma_i \in \{0, 1\}^n$, there is in fact a unique probability measure μ_d on these clopen sets such that $\mathcal{T} = \mathcal{T}_d$ and this can be computed as follows.

Suppose first that $\mathcal{T}(I(i)) = a_i$ for $i < 2$ and note that each $a_i \leq 1$ and $a_0 + a_1 \geq 1$ by the alternating property. If $\mathcal{T} = \mathcal{T}_d$, then we must have $d((0)) + d((2)) = a_0$ and $d((1)) + d((2)) = a_1$ and also $d((0)) + d((1)) + d((2)) = 1$, so that $d((2)) = a_0 + a_1 - 1$, $d((0)) = 1 - a_1$ and $d((1)) = 1 - a_0$. This will imply that $\mathcal{T}(I\tau) = \mathcal{T}_d(I(\tau))$ when $|\tau| = 1$. Now suppose that we have defined $d(\tau)$ and that τ is the code for a finite tree with elements $\sigma_0, \dots, \sigma_n = \sigma$ and thus $d(\tau \frown i)$ is giving the probability that σ will have one or both immediate successors. We proceed as above. Let $\mathcal{T}(I(\sigma \frown i)) = a_i \cdot \mathcal{T}(I(\sigma))$ for $i < 2$. Then as above $d(\tau \frown 2) = d(\tau) \cdot (a_0 + a_1 - 1)$ and $d(\tau \frown i) = d(\tau) \cdot (1 - a_i)$ for each i . \square

3 When is $\mathcal{T}(Q) = 0$?

In this section, we compute the capacity of a random closed set under certain probability measures. We construct a Π_1^0 class with measure zero but with positive capacity.

We say that $K \in \mathcal{C}$ is μ_d^* -random if x_K is Martin-Löf random with respect to the measure μ_d . (See [16] for details.)

Our next result shows that the \mathcal{T}_d capacity of a μ_d^* -random closed set depends on the particular measure.

Theorem 3.1. *Let d be the uniform measure with $b_0 = b_1 = b > 0$ and $b_2 = 1 - 2b > 0$ and let $\hat{b} = 1 - \frac{\sqrt{2}}{2}$. Then*

- (a) *If $b \geq \hat{b}$, then for any μ_d^* -random closed set R , $\mathcal{T}_d(R) = 0$.*
- (b) *If $b < \hat{b}$, then there is a μ_d^* -random closed set R with $\mathcal{T}_d(R) > 0$.*

Proof. Fix d as described above so that $d(\sigma \frown i) = d(\sigma) \cdot b$ and let $\mu^* = \mu_d^*$. We will compute the probability, given two closed sets Q and K , that $Q \cap K$ is nonempty. Here we define the usual product measure on the product space $\mathcal{C} \times \mathcal{C}$ of pairs (Q, K) of nonempty closed sets by letting $\mu^2(U_A \times U_B) = \mu^*(U_A) \cdot \mu^*(U_B)$ for arbitrary subsets A, B of $\{0, 1\}^n$.

Let

$$Q_n = \bigcup \{I(\sigma) : \sigma \in \{0, 1\}^n \text{ \& } Q \cap I(\sigma) \neq \emptyset\}$$

and similarly for K_n . Then $Q \cap K \neq \emptyset$ if and only if $Q_n \cap K_n \neq \emptyset$ for all n . Let p_n be the probability that $Q_n \cap K_n \neq \emptyset$ for two arbitrary closed sets K and Q , relative to our measure μ^* . It is immediate that $p_1 = 1 - 2b^2$, since $Q_1 \cap K_1 = \emptyset$ only when $Q_1 = I(i)$ and $K_1 = I(1 - i)$. Next we will determine the quadratic function f such that $p_{n+1} = f(p_n)$. There are 9 possible cases for Q_1 and K_1 , which break down into 4 distinct cases in the computation of p_{n+1} .

Case (i): As we have seen, $Q_1 \cap K_1 = \emptyset$ with probability $1 - 2b^2$.

Case (ii): There are two chances that $Q_1 = K_1 = I(i)$, each with probability b^2 so that $Q_{n+1} \cap K_{n+1} \neq \emptyset$ with (relative) probability p_n .

Case (iii): There are four chances where $Q_1 = 2^{\mathbb{N}}$ and $K_1 = I(i)$ or vice versa, each with probability $b \cdot (1 - 2b)$, so that once again $Q_{n+1} \cap K_{n+1} \neq \emptyset$ with relative probability p_n .

Case (iv): There is one chance that $Q_1 = K_1 = 2^{\mathbb{N}}$, with probability $(1 - 2b)^2$, in which case $Q_{n+1} \cap K_{n+1} \neq \emptyset$ with relative probability $1 - (1 - p_n)^2 = 2p_n - p_n^2$. This is because $Q_{n+1} \cap K_{n+1} = \emptyset$ if and only if both $Q_{n+1} \cap I(i) \cap K_{n+1} = \emptyset$ for both $i = 0$ and $i = 1$.

Adding these cases together, we see that

$$p_{n+1} = [2b^2 + 4b(1 - 2b)]p_n + (1 - 2b)^2(2p_n - p_n^2) = (2b^2 - 4b + 2)p_n - (1 - 4b + 4b^2)p_n^2.$$

Next we investigate the limit of the computable sequence $\langle p_n \rangle_{n \in \omega}$. Let $f(p) = (2b^2 - 4b + 2)p - (1 - 4b + 4b^2)p^2$. Note that $f(0) = 0$ and $f(1) = 1 - 4b^2 < 1$. It is easy to see that the fixed points of f are $p = 0$ and $p = \frac{2b^2 - 4b + 1}{(1 - 2b)^2}$. Note that since $b < \frac{1}{2}$, the denominator is not zero and hence is always positive.

Now consider the function $g(b) = 2b^2 - 4b + 1 = 2(b - 1)^2 - 1$, which has positive root \hat{b} and is decreasing for $0 \leq b \leq 1$. There are three cases to consider when comparing b with \hat{b} .

Case 1: If $b > \hat{b}$, then $g(b) < 0$ and hence the other fixed point of f is negative. Furthermore, $2b^2 - 4b + 2 < 1$ so that $f(p) < p$ for all $p > 0$. It follows that the sequence $\{p_n : n \in \mathbb{N}\}$ is decreasing with lower bound zero and hence must converge to a fixed point of f (since $p_{n+1} = f(p_n)$). Thus $\lim_n p_n = 0$.

Case 2: If $b = \hat{b}$, then $g(b) = 0$ and $f(p) = p - (4b - 1)p^2$, so that $p = 0$ is the unique fixed point of f . Furthermore, $4b - 1 = 3 - 2\sqrt{2} > 0$, so again $f(p) < p$ for all p . It follows again that $\lim_n p_n = 0$.

In these two cases, we can define a Martin-Löf test to prove that $T_d(R) = 0$ for any μ -random closed set R .

For each $m, n \in \omega$, let

$$B_m = \{(K, Q) : K_m \cap Q_m \neq \emptyset\},$$

so that $\mu^*(B_m) = p_m$ and let

$$A_{m,n} = \{Q : \mu^*(\{K : K_m \cap Q_m \neq \emptyset\}) \geq 2^{-n}\}.$$

Claim 3.2. For each m and n , $\mu^*(A_{m,n}) \leq 2^n \cdot p_m$.

Proof of Claim 3.2. Define the Borel measurable function $F_m : \mathcal{C} \times \mathcal{C} \rightarrow \{0, 1\}$ to be the characteristic function of B_m . Then

$$p_m = \mu^2(B_m) = \int_{Q \in \mathcal{C}} \int_{K \in \mathcal{C}} F(Q, K) dK dQ.$$

Now for fixed Q ,

$$\mu^*(\{K : K_m \cap Q_m \neq \emptyset\}) = \int_{K \in \mathcal{C}} F(Q, K) dK,$$

so that for $Q \in A_{m,n}$, we have $\int_{K \in \mathcal{C}} F(Q, K) dK \geq 2^{-n}$. It follows that

$$\begin{aligned} p_m &= \int_{Q \in \mathcal{C}} \int_{K \in \mathcal{C}} F(Q, K) dK dQ \geq \int_{Q \in A_{m,n}} \int_{K \in \mathcal{C}} F(Q, K) dK dQ \\ &\geq \int_{Q \in A_{m,n}} 2^{-n} dQ = 2^{-n} \mu^*(A_{m,n}). \end{aligned}$$

Multiplying both sides by 2^n completes the proof of Claim 3.2. \square

Since the computable sequence $\langle p_n \rangle_{n \in \omega}$ converges to 0, there must be a computable subsequence m_0, m_1, \dots such that $p_{m_n} < 2^{-2n-1}$ for all n . We can now define our Martin-Löf test. Let

$$S_r = A_{m_r, r}$$

and let

$$V_n = \cup_{r > n} S_r.$$

It follows that

$$\mu^*(A_n) \leq 2^{n+1} \mu^*(B_{m_n}) < 2^{n+1} 2^{-2n-1} = 2^{-n}$$

and therefore

$$\mu^*(V_n) \leq \sum_{r > n} 2^{-r} = 2^{-n}$$

Now suppose that R is a random closed set. The sequence $\langle V_n \rangle_{n \in \omega}$ is a computable sequence of c.e. open sets with measure $\leq 2^{-n}$, so that there is some n such that $R \notin S_n$. Thus for all $r > n$, $\mu^*(\{K : K_{m_r} \cap R_{m_r} \neq \emptyset\}) < 2^{-r}$ and it follows that

$$\mu^*(\{K : K \cap R \neq \emptyset\}) = \lim_n \mu^*(\{K : K_{m_n} \cap R_{m_n} \neq \emptyset\}) = 0.$$

Thus $\mathcal{T}_d(R) = 0$, as desired.

Case 3: Finally, suppose that $b < \hat{b}$. Then $0 < 2b^2 - 4b + 1 < 1$, so that f has a positive fixed point $m_b = \frac{2b^2 - 4b + 1}{(1 - 2b)^2}$. It is clear that $f(p) > p$ for $0 < p < m_b$ and $f(p) < p$ for $m_b < p$. Furthermore, the function f has its maximum at $p = [\frac{1-b}{1-2b}]^2 > 1$, so that f is monotone increasing on $[0, 1]$ and hence $f(p) > f(m_b) = m_b$ whenever $p > m_b$. Observe that $p_0 = 1 > m_b$ and hence the sequence $\{p_n : n \in \mathbb{N}\}$ is decreasing with lower bound m_b . It follows that $\lim_n p_n = m_b > 0$.

Now $B = \{(Q, K) : Q \cap K \neq \emptyset\} = \cap_n B_n$ is the intersection of a decreasing sequence of sets and hence $\mu^2(B) = \lim_n \mu^2(B_n) = m_b > 0$.

Claim 3.3. $\mu^*(\{Q : \mu^*(\{K : K \cap Q \neq \emptyset\}) > 0\}) \geq m_b$.

Proof of Claim 3.3. Let $B = \{(K, Q) : K \cap Q \neq \emptyset\}$, let $A = \{Q : \mu^*(\{K : K \cap Q \neq \emptyset\}) > 0\}$ and suppose that $\mu^*(A) < m_b$. As in the proof of Claim 3.2, we have

$$m_b = \mu^2(B) = \int_{Q \in \mathcal{C}} \int_{K \in \mathcal{C}} F(Q, K) dK dQ.$$

For $Q \notin A$, we have $\int_{K \in Q} F(Q, K) dK = \mu^*(\{K : K \cap Q \neq \emptyset\}) = 0$, so that

$$m_b = \int_{Q \in A} \int_{K \in Q} F(Q, K) dK dQ \leq \int_{Q \in A} dQ = \mu^*(A),$$

which completes the proof of Claim 3.3. \square

Claim 3.4. $\{Q : \mathcal{T}_d(Q) \geq m_b\}$ has positive measure.

Proof of Claim 3.4. Recall that $T_d(Q) = \mu^*(\{K : Q \cap K \neq \emptyset\})$. Let $B = \{(K, Q) : K \cap Q \neq \emptyset\}$, let $A = \{Q : T_d(Q) \geq m_b\}$ and suppose that $\mu^*(A) = 0$. As in the proof of Claim 3.2, we have

$$m_b = \mu^2(B) = \int_{Q \in \mathcal{C}} T_d(Q) dQ.$$

Since $\mu^*(A) = 0$, it follows that for any $B \subseteq \mathcal{C}$, we have

$$\int_{Q \in B} T_d(Q) dQ \leq m_b \mu^*(B).$$

Furthermore, $T_d(Q) < m_b$ for almost all Q , so there exists some P with $T_d(P) < m_b - \varepsilon$ for some positive ε . This means that for some n , $\mu^*(\{K : P_n \cap K_n \neq \emptyset\}) < m_b - \varepsilon$. Then for any closed set Q with $Q_n = P_n$, we have $T_d(Q) < m_b - \varepsilon$. But $E = \{Q : Q_n = P_n\}$ has positive measure, say $\delta > 0$. Then we have

$$\begin{aligned} m_b &= \int_{Q \in \mathcal{C}} T_d(Q) dQ = \int_{Q \in E} T_d(Q) dQ + \int_{Q \notin E} T_d(Q) dQ \\ &\leq \delta(m_b - \varepsilon) + (1 - \delta)m_b = m_b - \varepsilon\delta < m_b. \end{aligned}$$

This contradiction demonstrates Claim 3.4. \square

Since the set of μ^* -random closed sets has measure one, there must be a random closed set R such that $\mathcal{T}_d(R) \geq m_b$ and in particular, there is a μ^* -random closed set with positive capacity. \square

Thus for certain measures, there exists a random closed set with measure zero but with positive capacity. For the standard measure, a random closed set has capacity zero.

Corollary 3.5. *Let d be the uniform measure with $b_0 = b_1 = b_2 = \frac{1}{3}$. Then for any μ_d^* -random closed set R , $\mathcal{T}_d(R) = 0$.*

A random closed set may not be effectively closed. But we can also construct an effectively closed set with measure zero and with positive capacity.

Theorem 3.6. *For the regular measure μ_d with $b = b_1 = b_2$, there is a Π_1^0 class Q with Lebesgue measure zero and positive capacity $\mathcal{T}_d(Q)$.*

Proof. First let us compute the capacity of $X_n = \{x : x(n) = 0\}$. For $n = 0$, we have $\mathcal{T}_d(X_0) = 1 - b$. That is, Q meets X_0 if and only if $Q_0 = I(0)$ (which occurs with probability b), or $Q_0 = 2^{\mathbb{N}}$ (which occurs with probability $1 - 2b$). Now the probability $\mathcal{T}_d(X_{n+1})$ that an arbitrary closed set K meets X_{n+1} may be calculated in two distinct cases. As in the proof of Theorem 3.5, let

$$K_n = \bigcup \{I(\sigma) : \sigma \in \{0, 1\}^n \text{ \& } K \cap I(\sigma) \neq \emptyset\}$$

Case I If $K_0 = 2^{\mathbb{N}}$, then $\mathcal{T}_d(X_{n+1}) = 1 - (1 - \mathcal{T}_d(X_n))^2$.

Case II If $K_0 = I((i))$ for some $i < 2$, then $\mathcal{T}_d(X_{n+1}) = \mathcal{T}_d(X_n)$.

It follows that

$$\begin{aligned} \mathcal{T}_d(X_{n+1}) &= 2b\mathcal{T}_d(X_n) + (1 - 2b)(2\mathcal{T}_d(X_n) - (\mathcal{T}_d(X_n))^2) \\ &= (2 - 2b)\mathcal{T}_d(X_n) - (1 - 2b)(\mathcal{T}_d(X_n))^2 \end{aligned}$$

Now consider the function $f(p) = (2 - 2b)p - (1 - 2b)p^2$, where $0 < b < \frac{1}{2}$. This function has the properties that $f(0) = 0$, $f(1) = 1$ and $f(p) > p$ for $0 < p < 1$. Since $\mathcal{T}_d(X_{n+1}) = f(\mathcal{T}_d(X_n))$, it follows that $\lim_n \mathcal{T}_d(X_n) = 1$ and is the limit of a computable sequence.

For any $\sigma = (n_0, n_1, \dots, n_k)$, with $n_0 < n_1 < \dots < n_k$, similarly define $X_\sigma = \{x : (\forall i < k)x(n_i) = 0\}$. A similar argument to that above shows that $\lim_n \mathcal{T}_d(X_{\sigma \smallfrown n}) / \mathcal{T}_d(X_\sigma) = 1$.

Now consider the decreasing sequence $c_k = \frac{2^{k+1}+1}{2^{k+2}}$ with limit $\frac{1}{2}$. Choose $n = n_0$ such that $\mathcal{T}_d(X_n) \geq \frac{3}{4} = c_0$ and for each k , choose $n = n_{k+1}$ such that $\mathcal{T}_d(X_{(n_0, \dots, n_k, n)}) \geq c_{k+1}$. This can be done since $c_{k+1} < c_k$. Finally, let $Q = \bigcap_k X_{(n_0, \dots, n_k)}$. Then $\mathcal{T}_d(Q) = \lim_k \mathcal{T}_d(X_{(n_0, \dots, n_k)}) \geq \lim_k c_k = \frac{1}{2}$. \square

4 Conclusions and Future Research

In this paper, we have established a connection between measure and capacity for the space \mathcal{C} of closed subsets of $2^{\mathbb{N}}$. We showed that for a computable measure μ^* , a computable capacity may be defined by letting $\mathcal{T}(Q)$ be the measure of the family of closed sets K which have nonempty intersection with Q . We have proved an effective version of the Choquet's theorem by showing that every computable capacity may be obtained from a computable measure in this way.

For the uniform measure μ under which a node σ in T has exactly one immediate extension $\sigma \smallfrown i$ with probability b for $i = 0, 1$ (and hence σ has both extensions with probability $1 - 2b$), we have established conditions on b that characterize when the capacity of a random closed set equals zero or is > 0 . We have also constructed for each such measure an effectively closed set with positive capacity and with Lebesgue measure zero.

In future work, we plan to extend our results to more general measures where for each string $\sigma \in T_Q$, the probability that $\sigma \smallfrown i \in T_Q$ depends on σ . For example, such a measure on the space of closed sets may be defined by making the probability that both extensions $\sigma \smallfrown i$ of a node $\sigma \in T$ belong to T equal to $1 - \frac{2}{n}$ and the probability that just one extension belongs to T equal to $\frac{1}{n}$, where $n = |\sigma|$.

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